fluence on the solution. Both the heat balance integral method and the Galerkin method result in one solution. Of the three methods which lead to a unique value for the penetration depth under the assumption of the parabolic temperature profile, the Biot method results in the best agreement with the exact solution.

Constant flux boundary condition

The results for the flux boundary condition are compared in Table 2, where the penetration depth and surface temperature are, respectively,

$$\delta = C_1(\alpha t)^{1/2},\ \theta_s = C_2(F/k)(\alpha t)^{1/2} \qquad (14)$$

In obtaining the entries for the collocation and heat balance methods, θ_s was related to δ directly through the flux condition

$$F = -k(\partial\theta/\partial x) \qquad (15)$$

The solution by Galerkin's method was determined using both θ_s and δ separately, as weighting variables and therefore two different values of C_1 and C_2 are listed in Table 2. Each of the residual techniques transformed the heat equation into a first-order ordinary differential equation which was solved for the unknown generalized coordinate. However, the Biot method resulted in two ordinary differential equations: one given by the Lagrangian equation, Eq. (8), and one from the constraint equation, Eq. (10). As with the Galerkin technique, the solution was obtained by using either δ or θ_s as the independent generalized coordinate. Note, however, a somewhat different approach can be taken in the variational formulation if instead of Eq. (11), the profile is taken as

$$\theta = A(t)(1 - x/\delta)^2 \qquad (16)$$

where $A(t)$ is a function of the heat flux. By using the constraint equation directly, $\dot{H} = F$ at $x = 0$, θ_s is eliminated from the analysis, and a single differential equation for δ must be solved. This result is indicated as the "flux condition" in Table 2. Also shown in Table 2 are the numerical values which result when temperature profiles other than parabolic are assumed.[7] Table 2 shows that the Biot method gives the best agreement with the exact solution for surface temperature.

Conclusions

For the two simple problems discussed in this Note, each approximate method involved about the same amount of analytical effort. However, for nonlinear problems in which the thermal properties are temperature-dependent and the boundary conditions involve both convection and radiation which may change with time, the Biot method is probably the most flexible to apply. In this regard, Ref. (11) presents results based on the Biot formulation for the one-dimensional heat conduction in flat plates, which compare favorably with those from a finite-difference technique.

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Tube-Wall Fin Effects in Spacecraft Radiators

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Nomenclature

h	=	convection heat-transfer coefficient, Btu/hr-ft²-°F
k	=	thermal conductivity, Btu/hr-ft-°F
L_f	=	fin length, ft
Nu	=	$2R\int_0^\pi q_w(\theta)d\theta/(\pi k_f)$, dimensionless
Q	=	heat-transfer rate, per unit length of radiator, Btu/hr-ft
$q_w(\theta)$	=	heat flux to wall, Btu/hr-ft²
R	=	tube radius, ft
r	=	radial coordinate, made dimensionless with R
T	=	temperature, °F
$\bar{T}_f$	=	$\left(\int_0^\pi\int_0^R T_f w r dr d\theta\right) / \left(\int_0^\pi\int_0^R w r dr d\theta\right)$
t	=	tube-wall thickness, ft
w	=	fluid velocity, made dimensionless with $\bar{w}$
$\bar{w}$	=	average fluid velocity, ft/hr
z	=	axial coordinate, made dimensionless with $\alpha/(\bar{w}R^2)$
α	=	thermal diffusivity of fluid, ft²/hr
θ	=	circumferential angle, deg
ν	=	$k_fR/(k_wt)$
η	=	fin effectiveness

Subscripts

f,w	=	fluid and wall, respectively
w_1,w_2	=	tube wall region $\theta \leq \theta_1$ and $\theta_1 < \theta$, respectively
0	=	values calculated with isothermal wall

Introduction

PREVIOUS analyses of spacecraft radiator performances[1–5] are based in part on the assumption that heat transfer between the fluid (at T_f) and the tube wall (at T_w) is governed by $Q = h(2R)(\bar{T}_f - T_w)$ where h is calculated from theories derived on the assumption that T_w is uniform in the circumferential direction. This Note presents 1) a procedure for including the effects of nonuniform wall temperatures and 2) a complete solution for the extreme case of nonuniform wall temperature in laminar flow.

Figure 1 represents a cross-section of one tube, or fluid passage, and its associated fin. A radiator is composed of many such tube-fin assemblies arranged in series and parallel circuits. The radiating fin frequently also serves as the external structural skin of the vehicle. A complete analysis

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of the combined convection-conduction-radiation problem represented by Fig. 1 can be performed only by numerically solving the governing difference equations with the help of a computer. A more useful analysis can be performed if the problem is considered in two segments: 1) the fluid and tube system, and 2) the fin-tube bond and radiant-fin system; a line between points A and C in Fig. 1 is considered an isothermal interface between the two segments. Such a model is implied in the analyses of Refs. 3–5, the authors assuming that the first system is adequately modeled if the tube wall is considered circumferentially isothermal. A more realistic temperature distribution might appear as shown in Fig. 1. If the temperature is assumed uniform in the radial direction through the tube wall, as in conventional fin analyses, T_w is governed by the differential equation,[6]

$$d^2T_w/d\theta^2 = \nu \partial T_f/\partial r|_{r=1} \tag{1}$$

where the fluid temperature is given by[7]

$$w(\partial T_f/\partial z) = (1/r)(\partial/\partial r)(r\partial T_f/\partial r) + r^{-2}\partial^2 T_f/\partial\theta^2 \tag{2}$$

In Eqs. (1) and (2), longitudinal conduction is neglected. In laminar flow the velocity distribution is given by

$$w = 2(1 - r^2) \tag{3}$$

For the $T_w(\theta)$ shown in Fig. 1, Eq. (1) is most readily solved in two regions, $\theta \leq \theta_1$ and $\theta_1 \leq \theta \leq \pi$. Let the solutions in these two regions be designated T_{w_1} and T_{w_2}, respectively. Then the governing boundary conditions are as follows.

1) $T_{w_2}(\theta)$ is symmetric about $\theta = \pi$:

$$\partial T_{w_2}/\partial\theta = 0 \text{ at } \theta = \pi \tag{4}$$

2) T_w is independent of θ for $\theta \leq \theta_1$:

$$T_{w_2} = T_{w_1} \text{ at } \theta = \theta_1 \text{ and } T_{w_1} = Nu\cdot z \tag{5}$$

3) T_f (everywhere finite) is continuous at the wall:

$$T_f(R,\theta,z) = T_w(\theta,z) \tag{6}$$

4) q_w is independent of axial distance, z:

$$\partial T/\partial z = Nu = \text{const} \tag{7}$$

The solution to Eq. (2) can be obtained by superposition:

$$T_f(r,z,\theta) = Nu\cdot z + 2Nu\left[\frac{r^2}{4} - \frac{r^4}{16} + c_0\right] + \sum_{n=1}^{\infty} c_n r^n \cos n\theta \tag{8}$$

This equation can be substituted into Eq. (1), and two integrations over θ can be performed to get

$$T_{w_2}(\theta,z) = Nu\cdot z + \frac{\nu Nu}{2}\left[\frac{\theta^2}{2} - \pi\theta - \frac{\theta_1^2}{2} + \pi\theta_1\right] - \sum_{n=1}^{\infty}\frac{c_n}{n}(\cos n\theta - \cos n\theta_1) \tag{9}$$

Equations (4) and (8) were used to determine the integration constants. Equation (6) can be used to determine the c_n's:

$$\frac{\pi}{2}c_k = \frac{\nu Nu}{2}\left[-\frac{\pi^2}{3} - \frac{\theta_1^2}{2} + \pi\theta_1\right]\frac{\sin k\theta_1}{k} + \frac{\nu}{k}\sin k\theta_1 \sum_{n=1}^{\infty}\frac{c_n}{n}\cos n\theta_1 - \frac{\nu}{k}c_k\left(\frac{\pi - \theta_1}{2}\right) + \frac{\nu Nu}{k^2}\left(\frac{\pi-\theta_1}{2}\right) + \frac{\nu}{2}\sum_{\substack{n=1\\ n\neq k}}^{\infty}\left(\frac{c_n}{n} - \frac{\nu Nu}{n^2}\right)\left(\frac{\sin(k+n)\theta_1}{k+n} + \frac{\sin(k-n)\theta_1}{k-n}\right) \tag{10}$$

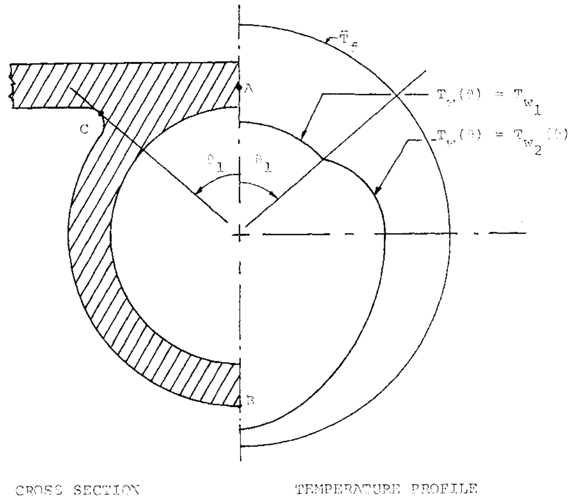

Fig. 1 Typical radiator cross section and associated tube-wall temperature distribution.

$$\pi c_0 = -\frac{3\pi}{16} + \frac{\nu}{4}\left[-\pi^2(\pi - \theta_1)/3 - \left(\frac{\theta_1^2}{2} - \pi\theta_1\right)(\pi - \theta_1)\right] + \frac{\nu}{2Nu}\sum_{n=1}^{\infty}\frac{c_n}{n}(\pi - \theta_1)\times \cos n\theta_1 + \frac{\nu}{4}\left[-\sum_{n=1}^{\infty}\frac{2}{n^3}\sin n\theta_1\right] + \frac{\nu}{2Nu}\sum_{n=1}^{\infty}\frac{c_n}{n^2}\sin n\theta_1 \tag{11}$$

Equation (10) represents a system of equations that must be solved simultaneously for the c_n's ($n = 1, 2, \ldots$). Once these c_n's are determined, c_0 can be determined from Eq. (11). With all of the c_n's specified, Eqs. (8, 10, and 11) define the complete temperature field. In solving Eq. (10), the summation must be terminated after a finite number of terms. The number of terms must be increased until adequate convergence is obtained.

In the spacecraft radiator problem, the quantity of interest is the heat-transfer rate from fluid to tube, $Nu = h(2R)(T_{w_1} - \bar{T}_f)/k_f$. Equations (3) and (8) can be inserted in the definition for $\bar{T}_f$ to yield $\bar{T}_f = Nu[z + 2c_0 + 7/48]$. Therefore $h(2R)/k_f = -[2c_0 + 7/48]^{-1}$. If h_0 is the heat-transfer coefficient for an isothermal wall, we can define

$$\eta \equiv h/h_0 = -11/(96c_0 + 7) \tag{12}$$

so we can calculate the heat rejection by the conventional equation, corrected for η

$$Q = h_0\eta\cdot 2\pi R(T_{w_1} - \bar{T}_f) \tag{13}$$

Solution for an Extreme Case

The effects of nonuniform T_w will be most severe when $\theta_1 = 0$. In this case, the solution to Eq. (10) can be obtained directly as $c_k/Nu = \nu/[k(k + \nu)]$ for $k = 1, 2, \ldots$ so that, by Eqs. (11) and (12), Formula 19.19 of Ref. 8, and some algebra

$$\eta = \left[1 + \frac{48\nu}{11}\sum_{k=1}^{\infty}\frac{1}{k(k+\nu)}\right]^{-1} \tag{14}$$

Eqs. (13) and (14) represent the exact solution to the subject problem in laminar flow. Values of η are presented in Table 1.

Approximate Solution

A simpler but less accurate solution to the problem just analyzed might be made if the interaction between the convection and conduction is neglected. The heat-transfer

Table 1 Comparison between approximate (η_0) and exact (η) equations for fin effect; $\theta_1 = 0$

$\nu \equiv k_f R / k_w t$	η	η_0	η/η_0
0.001	0.993	0.993	1.0
0.003	0.979	0.979	1.0
0.01	0.934	0.934	1.0
0.03	0.826	0.829	0.997
0.1	0.599	0.613	0.978
0.3	0.360	0.389	0.926
1.	0.1864	0.2155	0.865
3.	0.1111	0.1244	0.893
10.	0.0726	0.0682	1.065
30.	0.0543	0.0393	1.38
100.	0.0423	0.0216	1.96
1000.	0.0297	0.0068	4.36

coefficient can be evaluated from the usual theory, giving $h = h_0$. The fin effect of the tube wall can be incorporated into the analysis by use of the straight-fin effectiveness[6];

$$\eta_0 = [\tanh(h_0 L_f^2/k_w t)^{1/2}]/(h_0 L_f^2/k_w t)^{1/2} \tag{15}$$

For $\theta_1 = 0$, $L_f = \pi R$. Since[6] $h_0 = 48 k_f/(22R)$

$$\eta_0 = [\tanh(24\pi^2\nu/11)^{1/2}]/(24\pi^2\nu/11)^{1/2} \tag{16}$$

The values of η and η_0 are compared in Table 1. In practice, ν might range from 0.001 for dielectric fluids in small aluminum tubes to 100 for liquid metals in large stainless-steel tubes. In the latter case, the flow is more likely to be turbulent, so applicability of the laminar flow solution is probably limited to low values of ν, where $\eta_0 \simeq \eta$.

Conclusions

An analytical model has been developed to predict the effects of circumferential wall-temperature gradients in spacecraft radiator tubes. Exact solutions have been derived for the fin effectiveness of the tube wall in terms of an infinite system of linear equations. A convenient expression has been derived for the extreme case of point contact between tube wall and radiating fin. An approximate solution, based on neglecting the conduction-convection interaction, has been derived for the same analytical model; it compares favorably with the exact solution for $(k_f R/k_w t) < 3$, which should include most cases of interest in environmental-control systems. For cases in which a larger contact region must be considered, so that the fin length is less than that for point contact, the approximate solution should give even more accurate results.

Similar analyses might be conducted for turbulent flow, using the hypothesis that the circumferential eddy diffusivity equals the radial eddy diffusivity.[7]

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Payload Optimization Factors for Storage of Liquid Hydrogen in a Low-Gravity Environment

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Nomenclature

c_p	=	specific heat at constant pressure, Btu/lbm − °R
Gr	=	Grashoff number = $\lvert B_g L^3 T/v^2 \rvert$
Bo	=	Bond number = gravity forces/viscous forces = $L^2 P_g/\sigma$
T	=	temperature, °R
MR	=	engine mixture ratio, $\dot{w}_{ox}/\dot{w}_{fuel}$
I_{sp}	=	specific impulse, sec
g	=	acceleration level, ft/sec²
T	=	thrust, lbf
L	=	length, ft
W	=	weight, lb; w = flowrate, lb/sec
$\dot{w}$	=	flowrate, lb/min
P	=	saturation pressure; P_t = tank pressure, psia
k	=	thermal conductivity, Btu/ft-hr-°F
ρ	=	density, lb/ft³
σ	=	surface tension, lbf/ft
μ	=	viscosity, lbm/hr-ft
ν	=	kinematic viscosity, μ/p
APS	=	auxiliary propulsion system

Introduction

MANY of the Apollo application missions and post-Apollo studies require orbital storage of cryogenic propellants; among these are such missions as the Saturn V Synchronous orbit, 110-hr, Lunar Applications Spent Stage (LASS) mission, and numerous orbital docking and propellant transfer experiments. During these periods in orbit, it will be necessary to vent the excess tank pressures (P_t) caused by

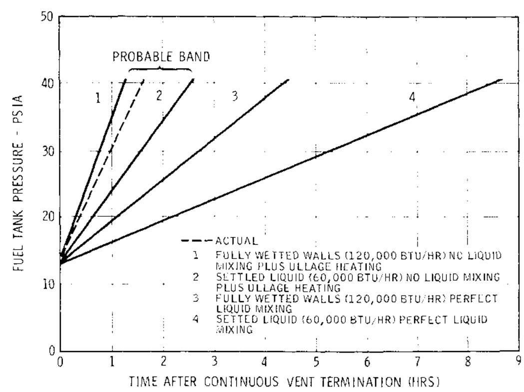

Fig. 1 Fuel tank (LH_2) pressure predictions (no venting).

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